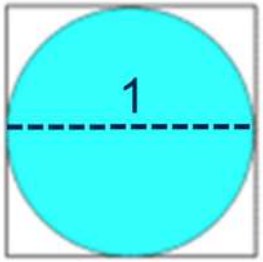
Four Short Problems Solutions

Circles

Diagram 1

The circle has a diameter of 1 as it is the same width as the square.

The radius of the circle therefore $= \frac{1}{2}$

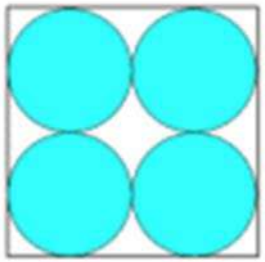


The area shaded = $\left(\frac{1}{2}\right)^2 \times \pi = \frac{1}{4}\pi$

Diagram 2

Each circle has a diameter of $\frac{1}{2}$ as 2 circles are the same width as the square.

The radius of a circle$= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

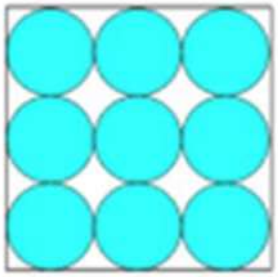


The area shaded = $\left(\frac{1}{4}\right)^2 \times \pi \times 4 = \frac{4}{16}\pi = \frac{1}{4}\pi$

Diagram 3

Each circle has a diameter of $\frac{1}{3}$ as 3 circles are the same width as the square.

The radius of a circle is $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

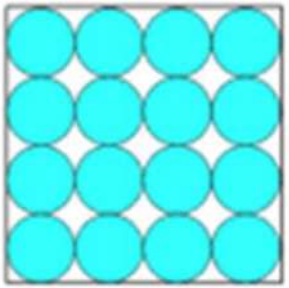


The area shaded = $\left(\frac{1}{6}\right)^2 \times \pi \times 9 = \frac{9}{36}\pi = \frac{1}{4}\pi$

Diagram 4

Each circle has a diameter of $\frac{1}{4}$ as 4 circles are the same width as the square.

The radius of a circle$= \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$



The area shaded = $\left(\frac{1}{8}\right)^2 \times \pi \times 16 = \frac{16}{64}\pi = \frac{1}{4}\pi$

So, all the shaded areas are equal!

Peaches

Question

A monkey has 75 peaches

Each day he keeps a fraction of the his peaches, gives some away and eats 1

These are the fractions he decided to keep.

$$\frac{1}{2} \quad \frac{1}{4} \quad \frac{3}{4} \quad \frac{3}{5} \quad \frac{5}{6} \quad \frac{11}{15}$$

In what order did he use the fractions so that he was left with just one peach at the end?

Possible solution

The monkey has 75 peaches at the start and decides to keep one of the following fractions $\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{3}{5}, \frac{5}{6}, \frac{11}{15}$

Firstly, if we look at the denominators there are only two that divide exactly into 75 and they are 5 and 15 so there are two possible options to start with

If we decide to go with $\frac{11}{15}$ the monkey would keep $\frac{11}{15}$ of 75 (55 peaches) give away 20 and eat 1 leaving 54 for Day 2

If we use the same process and look at the denominators of the remaining fractions only 2 and 6 divide into 54 exactly

If we decide to go with $\frac{5}{6}$ the monkey would keep $\frac{5}{6}$ of 54 (45 peaches) give 9 away and eat 1 leaving 44 peaches for Day 3

On Day 3 there are a few options but let us say go with $\frac{3}{4}$ and $\frac{3}{4}$ of 44 gives 33 peaches and 11 to give away and eating 1 gives 32 for day 4

On Day 4 we could say the monkey keeps $\frac{1}{2}$ (16 peaches) gives 16 away then eats one leaving 15 for Day 5

On Day 5 the monkey could keep $\frac{3}{5}$ of 15 (9 peaches) gives 6 away and eats 1 leaving 8 peaches for Day 6

On Day 6 the monkey keeps $\frac{1}{4}$ of 8 (2 peaches) gives 6 away and eats 1 leaving 1 at the end.

Now, obviously we made a choice at the start and we may have made a wrong choice which might mean we have to start again e.g. we could have chosen 3/5 ....

One approach here would be to use a tree diagram type approach when there are choices to consider.

Petrol Station

Question

Andrea's car has a petrol tank that holds 44 litres of petrol. She goes to the petrol station when her tank is a quarter full and fills it up until it is two thirds full.

How many litres of petrol does she put into the cars petrol tank?

Possible solution

The Petrol tank has a capacity of 44 litres.

It is already $\frac{1}{4}$ full – so that is $\frac{1}{4}$ of 44 which is 11 litres

After putting petrol in it is $\frac{2}{3}$ full

$\frac{2}{3}$ of 44 is $\frac{44 \times 2}{3} = \frac{88}{3} = 29\frac{1}{3}$ litres in tank after putting petrol in

originally it contained 11 litres, after putting petrol in it contains $29\frac{1}{3}$ litres

so, you have added $18\frac{1}{3}$ litres

Alternative solution

Originally it was $\frac{1}{4}$ full and after putting petrol in it is $\frac{2}{3}$ full

The fraction put in is the difference between $\frac{2}{3}$ and $\frac{1}{4}$

$\frac{2}{3} - \frac{1}{4} = \frac{8}{12} - \frac{3}{12} = \frac{5}{12}$

$\frac{5}{12}$ of 44 litres is $\frac{44 \times 5}{12} = \frac{220}{12}$

If we simplify, we get $\frac{55}{3} = 18\frac{1}{3}$ litres

Integers

Question

What is the integer x so that $\frac{x}{9}$ lies between $\frac{71}{7}$ and $\frac{113}{11}$

Possible solution

In this question the fraction we want must lie between $\frac{71}{7}$ and $\frac{113}{11}$

One way to consider this is to change them into mixed fractions so between $10\frac{1}{7}$ and $10\frac{3}{11}$

The answer must be $10\frac{?}{9}$

$\frac{1}{9}$ $\left(\frac{7}{63}\right)$ is less than $\frac{1}{7}\left(\frac{9}{63}\right)$ so it cannot be $\frac{1}{9}$

$\frac{3}{9}$ $\left(\frac{33}{99}\right)$ is more than $\frac{3}{11}$ $\left(\frac{27}{99}\right)$ so it cannot be $\frac{3}{9}$

So, we must have $10\frac{2}{9} = \frac{92}{9}$ and so x must be 92

Alternative solution

Halfway between 71 and 113 is 92

Halfway between 7 and 11 is 9

So $\frac{92}{9}$ must be between the 2 given fractions and $\frac{91}{9}$ would be too small and $\frac{93}{9}$ would be too big

So x must be 92

Fractions of 1000 Solutions

What is $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{4}{5}$ of $\frac{5}{6}$ of $\frac{6}{7}$ of $\frac{7}{8}$ of $\frac{8}{9}$ of $\frac{9}{10}$ of 1000 ?

If we think of this as a calculation this becomes

$$\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} \times \frac{5}{6} \times \frac{6}{7} \times \frac{7}{8} \times \frac{8}{9} \times \frac{9}{10} \times 1000$$

By writing the calculation out you can see there is some cancelling that can be done

$$\frac{1}{\cancel{2}} \times \frac{\cancel{2}}{\cancel{3}} \times \frac{\cancel{3}}{\cancel{4}} \times \frac{\cancel{4}}{\cancel{5}} \times \frac{\cancel{5}}{\cancel{6}} \times \frac{\cancel{6}}{\cancel{7}} \times \frac{\cancel{7}}{\cancel{8}} \times \frac{\cancel{8}}{\cancel{9}} \times \frac{\cancel{9}}{10} \times 1000$$

So the only calculation we need to do is

$$\frac{1}{10} \times 1000 = 100$$

Unit Fractions

$\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$ ✓ $\quad 1/2 = 1/10 + 1/20$ $\quad \frac{1}{3} = \frac{1}{4} + \frac{1}{12}$ ✓ $\quad 1/3 = 1/7 + 1/21$ $\quad ¼ = \frac{1}{5} + \frac{1}{20}$ ✓

So, you may have realised that for this to work one of the unit fractions must have a denominator 1 more than the original unit fraction on the left-hand side of the equation. The denominator on the other unit fraction would be the product of the other two.

For example, I could have $1/5 = 1/6+?$ The missing fraction would have denominator of 30 and we would get $1/5 = 1/6 + 1/30$

This leads to $1/n = 1/(n+1) + 1/(n(n+1))$

Some unit fractions can be made in more than one way

Here are some examples for $\frac{1}{6}$

$\frac{1}{6} = \frac{1}{9} + \frac{1}{18}$ $\quad \frac{1}{6} = \frac{1}{10} + \frac{1}{15}$ $\quad \frac{1}{6} = \frac{1}{12} + \frac{1}{12}$ (interesting)

$\frac{1}{8} = \frac{1}{9} + \frac{1}{72}$ $\quad \frac{1}{8} = \frac{1}{10} + \frac{1}{40}$ $\quad \frac{1}{8} = \frac{1}{12} + \frac{1}{24}$

$\frac{1}{6} = \frac{1}{11} + \frac{1}{?}$ $\quad \frac{1}{7} = \frac{1}{13} + \frac{1}{?}$ $\quad \frac{1}{3} = \frac{1}{5} + \frac{1}{?}$ (these cannot be done – what connects the numbers 11, 13 and 5?)

Unit fractions with denominators which are prime can only be written one way as the sum of two distinct unit fractions so, you can do $\frac{1}{7} = \frac{1}{8} + \frac{1}{56}$ but you cannot find another one for $\frac{1}{7}$